II. *On the Precession of the Equinoxes. By the Rev.* Abram Robertson, *M. A. F. R. S. Savilian Professor of Geometry in the University of Oxford.*

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PERHAPS the solution of no other problem, in natural philosophy, has so often baffled the attempts of mathematicians as that of determining the precession of the equinoxes, by the theory of gravity. The phenomenon itself was observed about one hundred and fifty years before the christian æra, but Sir ISAAC NEWTON was the first who endeavoured to estimate its magnitude by the true principles of motion, combined with the attractive influence of the sun and moon on the spheroidal figure of the earth. It has always been allowed, by those competent to judge, that his investigations relating to the subject evince the same transcendent abilities as are displayed in the other parts of his immortal work, THE MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY, but, for more than half a century past, it has been justly asserted that he made a mistake in his process, which rendered his conclusions erroneous.

Since the detection of this error, some of the most eminent mathematicians in Europe have attempted solutions of the problem. Their success has been various; but their investigations may be arranged under three general heads. Under the first of these may be placed such as lead to a wrong

conclusion, in consequence of a mistake committed in some part of the proceedings. The second head may be allotted to those in which the conclusions may be admitted as just, but rendered so by the counteraction of opposite errors. Such may be ranked under the third head as are conducted without error fatal to the conclusion, and in which the result is as near the truth as the subject seems to admit.

The authors of those investigations, of each of the three descriptions, are entitled to much praise. Their productions afford the most unquestionable proofs of great talents, great zeal, and great perseverance, exerted in the cultivation of science. The mistakes committed in those of the two first descriptions, and the obscurity and perplexity with which those of the third may be charged, are, in my opinion, to be attributed to the same cause, the uncultivated state of that particular department of the doctrine of motion, which constitutes the appropriate foundation for the solution of the problem. The department to which I allude is that of compound rotatory motion.

In consequence of this persuasion I have, in the first nine of the following articles, endeavoured to investigate the primary properties of compound rotatory motion from clear and unexceptionable principles. The disturbing solar force on the spheroidal figure of the earth is then calculated, and the angular velocity which it produces is afterwards compared with that of the diurnal revolution, by means of the properties of rotatory motion previously demonstrated. The quantity of annual precession is then calculated in the usual way, and also that of nutation, as far as they are produced by the disturbing force of the sun.

1. Let C, (Plate II. Fig. 1.) be the centre of two circular arcs AB, EF, which are the measures of the angles ACB, ECF; and let CB cut EF in D. Then, as the sectors ACB, ECD are similar, $CA : CE :: AB : ED = \frac{CE \times AB}{CA}$. But (Eu. 33. VI.) $ED : EF ::$ the angle ACB : the angle ECF; and therefore $\frac{CE \times AB}{CA} : EF :: < ACB : < ECF$. Consequently $CE \times AB \times < ECF = CA \times EF \times < ACB$; and therefore $AB : EF :: CA \times < ACB : CE \times < ECF$.

2. Let ACB, GEF (Fig. 2.) be two angles, of which the arcs AB, GF are the measures; and the radii CA, EG not being necessarily equal, let the sines BK, FQ be equal to one another. Let BH, FM be tangents to the curves; and let HD, MN be parallel to CA, EG respectively, and meet BK, FQ in D, N, as represented. Then as CBH, BDH are right angles, the triangles CBK, BHD are equiangular, and $CB : BK :: BH : HD$ or its equal KL, if HL be drawn parallel to BK, and meet CA in L. Consequently $BK = \frac{CB \times KL}{BH}$. For the same reasons, if MP be parallel to FQ and meet EG in P, $FQ = \frac{EF \times QP}{FM}$; and therefore as BK, FQ are, by hypothesis, equal, $\frac{CB \times KL}{BH} = \frac{EF \times QP}{FM}$. Hence $CB \times KL \times FM = EF \times QP \times BH$, and $BH : FM :: CB \times KL : EF \times QP$.

3. If, therefore, we suppose straight lines CH, EM to be drawn, and that the angles BCH, FEM are indefinitely small, and generated in the same time by the revolution of CB, EF respectively, then BH, FM may be considered as circular arcs, and by article 1, $BH : FM :: CB \times < BCH : EF \times < FEM$. Hence by article 2, (and 11. V.) $CB \times < BCH : EF \times < FEM :: CB \times KL : EF \times QP$, and therefore $< BCH : < FEM :: KL$

: QP. Consequently, during the generation of the angles BCH, FEM, the angular velocity of CB is to the angular velocity of EF as KL to QP.

4. If ACB (Fig. 3.) be any plane angle, and from any point E, EF be drawn perpendicular to AC and EH perpendicular to CB, then the angle FEH is equal to the angle ACB. For let HK be drawn perpendicular to AC, and let EH be produced to G. First let ACB be an obtuse angle, and then $< ACB = < KHC + < HKC + < KHC + < CHG = < KHG = (29. I.) < FEH$. Secondly, ACB being an acute angle, the right angle $GHC = < GHK + < KHC = < KHC + < HCK = (15. I.) < KHC + < ACB$. Consequently the angle $ACB = < GHK = < FEH$.

5. Let G (Fig. 4.) be the centre of gravity of a body, and AB, DC two axes passing through G; and, while the body revolves round AB, let AB and consequently the whole body revolve round DC, the periodical times of these revolutions not being necessarily equal; it is required to determine the direction and angular velocity with which any particle of the body revolves in consequence of this compound motion.

Suppose the simple motion of the body about AB to be such, that during the revolution the parts towards D from AB would rise above the plane, on which the figure is drawn, and the parts towards C sink below it. And suppose the simple motion about DC to be such, that during the revolution the parts towards A from DC would rise above the plane of the figure, and those towards B sink below it. Let P (Fig. 5.) be a particle of the body, above the plane, and let PR be a perpendicular to the said plane.* With the centre G suppose

* The axis DC, and the line RM are intentionally omitted in Fig. 5, with a view to prevent confusion in the figure.

a spherical superficies to be described, passing through P, and let ADBC be the great circle of this sphere in the plane on which the whole is represented. Let the straight lines EF, HK pass through R, and be perpendicular to AB, DC respectively; and let EPF, HPK be lesser circles of the sphere, EF being a diameter of the one, and HK a diameter of the other. Then it is evident, that by the simple motion of the body about AB only the particle P would move in the circumference FPE; and by the simple motion of the body about DC only, P would move in the circumference HPK. Let the indefinitely small arcs P*s*, P*q* be those which P would describe in equal times with the revolutions about AB, DC separately, and let the parallelogram P*qps* be completed on the spherical superficies. Then it is evident, from the composition of motion, that the direction and velocity of P, in consequence of the compound motion, is as the diagonal P*p* of the parallelogram P*qps*.

Let R*r*MN be the orthographical projection of P*qps* on the plane ADBC, and then as PR is perpendicular to this plane, it is evident that R*r*MN is a parallelogram, and that its diagonal RM is the direction and velocity of P in the projection, in consequence of the compound motion. It therefore follows, from article 3, as RN is the angular velocity about the axis AB, and R*r* that about the axis DC, that RM is the angular velocity about the axis, round which the body is caused to revolve by the compound motion.

6. The same things being supposed, and the parallelogram R*r*MN being the same in Fig. 4 as in Fig. 5, let RM produced meet the circumference in L and Q, and the diameter TGS, at right angles to LQ, is the axis sought.

The same axis may be obtained in the following manner.

In GB take GV equal to RN; and in GC take GW equal to R*r*, and VW being drawn it will be parallel to the axis TS. For as NR is perpendicular to AB, and R*r* or NM to DC, by article 4, the angle RNM, or (34. I.) its equal R*r*M, is equal to the angle VGW. Also, on account of the equals, VG : GW : : M*r* : *r*R, and therefore (6. VI.) the angles *r*RM, GWV are equal. Let TS meet HK in O, and LQ in I; and let DC meet HK in X. Then as the angle OIR is equal to the angle OXG, each being a right one, and as the angles IOR, XOG are equal, the angle IRO, or MR*r* is equal to the angle OGX. Consequently the alternate angles XGO, GWV are equal, and therefore TS, VW are parallel. Hence it is evident that if the axes AB, DC, and also GV, GW the angular velocities round them be given, the axis TS is easily found, being parallel to VW. It is proper to observe that GV, GW are to be set off on that side of TS towards which the body is moving, in consequence of the revolutions round DC, AB.

7. From the last article it is evident that VW is equal to RM, and consequently equal to the angular velocity, with which the body revolves about the axis TS. If therefore CGB be a right angle, then the angular velocity $VW = \sqrt{VG^2 + GW^2}$. In other cases the value of VW may be easily calculated by plane Trigonometry.

8. It is to be remarked, for the sake of precision, that the linear velocity, of any point, is as the angular velocity multiplied into the radius of the circle in whose circumference it revolves. Thus the linear velocity P*s* (Fig. 5.) of the point P in the circumference FPE, is as its angular velocity in the same, multiplied into the radius of the circle FPE, as is evi-

dent from article 3. In the following articles, linear velocity is meant when no adjective is annexed to the word velocity.

9. Let AB, DC (Fig. 6.) be the two axes about which separately the body would revolve, as stated in article 5, and let TS be the axis about which it revolves, in consequence of a combination of these two revolutions. Let TE be at right angles to AB, and meet it in H, and let TF be at right angles to DC and meet it in K; and let GV, GW be the angular velocities about AB, DC, as in the preceding articles. Then it follows, from the last article, that the velocity of the point T, by the revolution about the axis AB only, is equal to $GV \times HT$. And as this velocity is in the direction of a tangent at T to the circle of which TE is a diameter, and as this circle is perpendicular to the plane ADBC, the direction of this velocity is evidently perpendicular to the plane ADBC. The direction of this velocity of the point T is also upwards from the plane of the figure, agreeable to the statement in article 5. Again, by the revolution about the axis DC only, the velocity of T is equal to $GW \times KT$, and, for the foregoing reasons, the direction of this velocity of the point T is perpendicularly downwards below the plane, according to article 5. Now as TS is the axis about which the body revolves, in consequence of the combined revolutions about AB, DC, every point in TS is rendered quiescent by the compound motions. It is therefore evident that $GV \times HT = GW \times KT$.

10. The revolutions about DC, AB may be supposed to be caused by instantaneous impulses at A and D, made at the same time, or at different times; or they may be supposed to be occasioned by the agency of constant forces, like that of

gravity. For if the causes be adequate to the production of the same velocities, taken separately, and in the same directions, the velocity and direction of a particle will be the same from their combined influence upon it, whether these causes be impulses or constant forces.

As the body is understood to be in free space, if the causes of the revolutions, taken separately, be instantaneous impulses, and made at the same time, immediately after their agency the body will revolve about the axis TS, and it will continue so to revolve with an uniform velocity. If whilst the body is revolving with an uniform velocity about the axis DC, a constant force begin to act at D, so as to cause a tendency to revolution about AB, as stated in article 5, and continue afterwards to act at T, the pole of the new axis, from a combination of the constant agency at the new pole and the uniform velocity about DC, the axis TS will incessantly shift its position.

Such exactly are the circumstances to which the earth is subject as to the production of the precession of the equinoxes. At the vernal equinox, for instance, a straight line drawn from the centre of the sun to that of the earth is in the plane of the equator, and therefore, as equal portions of the protuberant matter of the earth are above and below the ecliptic, the attractive power of the sun has no tendency to alter the position of the equator. But, in consequence of the earth's motion in its orbit, it very soon after the equinox presents a different position of the equator to the sun. The equilibrium of the protuberant parts of the earth, above and below the ecliptic, and towards the sun, is then done away,

and the attraction of the sun on that side, where the greatest quantity of protuberant matter is, tends to bring down the equator into the ecliptic, or to cause the earth to revolve about a diameter of the equator. This attractive influence of the sun gradually increases a little till the summer solstice; it then gradually decreases in the same degree till the autumnal equinox, when it vanishes. From the autumnal equinox to the winter solstice it again gradually increases a little; and it then gradually decreases in the same degree till the vernal equinox, when it again vanishes. This recurrence and continuance of action is annually repeated.

Similar observations apply to the attraction of the moon on the protuberant parts of the earth. When a straight line drawn from her centre to that of the earth is in the plane of the equator, the attractive influence of the moon has no tendency to change the position of the equator, but in other situations, the attraction of the moon tends to bring the equator of the earth into the plane of the moon's orbit, or causes the earth to move round a diameter of the equator. The recurrences of the moon's action on the protuberant parts of the earth, and the times of their continuing, are repeated every month.

These effects of the sun and moon are to be considered separately; and for the reasons already stated, each of the actions, combined with the diurnal revolution of the earth, may be considered as a particular case of compound rotatory motion. It is needless, however, after investigating the effects of the sun's action, and expressing them in general formulæ, to go over the same steps for ascertaining those of the moon,

as they may be inferred from the former, after making due allowance for the different circumstances under which these two remote bodies act on the protuberant parts of the earth.

I now proceed to estimate the force with which the sun tends to cause the earth to revolve about a diameter of the equator.

11. Let S be the centre of the sun (Fig. 7.) C that of the earth; P, L the poles, PL the axis; and let a plane passing through SC, PL cut the earth in the meridian PEAQB. Let EQ be the diameter of the equator, and let DF, the diameter of the spheroid in the plane SPCL, be at right angles to SC. Let SC cut the meridian EPQL in A, B; and G being supposed a particle of matter in this meridian, let GH parallel to SC meet DF in H, and let SG be drawn. Let M be the quantity of matter in the sun, or its absolute attracting power, and then $\frac{M}{SG^2}$ is its force upon the particle G, in the direction SG, and $\frac{M}{SC^2}$ is its force upon a particle at C, in the direction SC. But a force whose power and direction is as GS is equal to a force whose power and direction is as GC, together with a force whose power and direction is as CS; and as the force whose power and direction is as GC, is directed to the centre it has no tendency to alter the position of the axis PL, and therefore may be neglected in the present enquiry. Now, by Mechanics, $SG : SC :: \frac{M}{SG^2} : \frac{M \times SC}{SG^3}$ = the force of the sun on the particle G, in the direction CS or HG. Now as the distance of the sun from the earth is indefinitely great when compared to the diameter DF, its force on any particle in DF is equal to its force on a particle at C, and therefore the sun's force on a particle at H is as $\frac{M}{SC^2}$. Consequently, as the sun's

force on the particle G, in the same direction, is as $\frac{M \times SC}{SG^3}$, the disturbing force of the sun, by its action on the particle G is $\frac{M \times SC}{SG^3} - \frac{M}{SC^2}$, or $\frac{M \times SC}{\overline{SC - GH}|^3} - \frac{M}{SC^2}$, for SG may be considered as equal to SC — GH. But as SC is indefinitely great with respect to GH, $\frac{SC}{\overline{SC - GH}|^3}$, by actual division may be considered as equal to $\frac{1}{SC^2} + \frac{3GH}{SC^3}$, and therefore the disturbing force on the particle at G is $\frac{3M \times GH}{SC^3}$.

Let K be a particle in the meridian, but on the opposite side of DF to that on which G is situated. Let KN, parallel to SC, meet DF in N; and suppose SK to be drawn. Then the force of the sun on K being $\frac{M}{SK^2}$, for the same reasons as before, its force upon it in the direction of SC or KN is $\frac{M \times SC}{\overline{SC + KN}|^3}$, and after a reduction similar to the foregoing, the sun's disturbing force on K is $-\frac{3M \times KN}{SC^3}$.

Hence it is evident, supposing M and SC to be constant, that the disturbing force of the sun on any particle in the meridian PELQ is as the distance of the particle from DF; and that the sign of the force in the half DAF nearest to S is positive, but the sign of the force in the other half DBF is negative. This difference of the signs indicates that the particles on the opposite sides of DF have a directly opposite tendency, as to direction, in affecting the position of the axis PL, or equator EQ; and the same is evident from the following considerations. As the disturbing force is as its distance from DF, it has no effect on particles in DF, and therefore the inertia * of

* By this expression that part of the inertia is meant which opposes the disturbing force of the sun; and the same is to be understood in the following expressions.

particles in DF may be considered as equal to the sun's disturbing force, on the principle of action and reaction being equal as to magnitude, but directly contrary as to direction. But on the side of DF nearest to S the disturbing force is greater than the inertia of any particle G, and it therefore urges the particle from DF towards S, by a pressure whose direction and power is as HG. On the side of DF opposite to S the disturbing force is less than the inertia of any particle K, and therefore the inertia of K opposes the disturbing force of the sun by a pressure whose direction is from N towards K, and whose power is as NK.

12. As, by the nature of the spheroid, PELQ is an ellipse, let GK be the diameter conjugate to DF, and let VI, parallel to DF, meet it in T, and AB in R; and then VI is bisected in T. Let RI be bisected in v, and let w, q be two points in RI, equally distant from R and I respectively. Let $a =$ EC, and $d =$ the disturbing force of the sun at the distance of EC from DF. Then by the preceding article, $a : \text{RC} :: d : \frac{d}{a} \times \text{RC}$ = the force at R, or at any point in VI, as any two points in VI are equally distant from DF. Now it is evident that the disturbing force on a particle at R, or on any particle in AC, has no power to turn the ellipse about C; but the force on a particle at w tends to turn the ellipse about the centre, for it is applied at the end of the lever Rw. Consequently, by what has been already proved in this article, and by the property of the lever, the force on w to turn the ellipse is $\frac{d}{a} \times \text{RC} \times \text{R}w$. For the same reasons, the force on q to turn the ellipse is $\frac{d}{a} \times \text{RC} \times \text{R}q$, and therefore the force on w and q combined, to turn the ellipse, is $\frac{d}{a} \times \text{RC} \times \text{RI}$, for $\text{R}w + \text{R}q = \text{RI}$.

Hence, as Rv or $\frac{\text{RI}}{2}$ expresses the number of particles in Rv or vI, it follows that the force of all the particles in RI, to turn the ellipse, is $\frac{d}{a} \times \text{RC} \times \text{RI} \times \frac{\text{RI}}{2} = \frac{d}{2a} \times \text{RC} \times \text{RI}^2$. In the same way it may be proved that the force of all the particles in VR, to turn the ellipse, is $\frac{d}{2a} \times \text{RC} \times \text{RV}^2$. But the force of the particles in RI tends to turn the ellipse upwards in the direction FAD, and the force of the particles in RV tends to turn it downwards in the contrary direction DAF. The force of all the particles in VI, therefore, to turn the ellipse, is $\frac{d}{2a} \times \text{RC} \times \overline{\text{RV}^2 - \text{RI}^2}$. But as TV is half the sum of RV, RI, it follows that RT is half their difference, and therefore $\text{RV}^2 - \text{RI}^2 = \text{VI} \times 2\text{RT}$. Consequently the force of the particles in VI, to turn the ellipse, is $\frac{d}{2a} \times \text{RC} \times \text{VI} \times 2\text{RT} = \frac{d}{a} \times \text{RC} \times \text{VI} \times \text{RT}$.

13. Let $c = \text{CG}$, $b = \text{CD}$, $f = \text{GH}$, $g = \text{CH}$, $y = \text{CR}$, and $x = \text{CT}$. Then by similar triangles, $c : f :: x : y = \frac{fx}{c}$; and $c : g :: x : \frac{gx}{c} = \text{RT}$. Also, by the property of the ellipse, $c^2 : b^2 :: \text{GT} \times \text{TK} : \text{TV}^2 :: \overline{c + x} \times \overline{c - x} : \text{TV}^2 :: c^2 - x^2 : \text{TV}^2$. Consequently $\text{TV} = \frac{b}{c}\sqrt{c^2 - x^2}$, and $\text{VI} = \frac{2b}{c}\sqrt{c^2 - x^2}$, and the force of the particles in VI, to turn the ellipse, is $\frac{d}{a} \times y \times \frac{2b}{c}\sqrt{c^2 - x^2} \times \frac{gx}{c} = \frac{2bdfgx^2}{ac^3}\sqrt{c^2 - x^2}$, by putting $\frac{fx}{c}$ for y. The fluxion of this force is therefore $\frac{2bdfgx^2}{ac^3}\sqrt{c^2 - x^2} \times \dot{y} = \frac{2bdfgx^2}{ac^3}\sqrt{c^2 - x^2} \times \frac{f\dot{x}}{c} = \frac{2bdffgx^2\dot{x}}{ac^4}\sqrt{c^2 - x^2}$, as $\frac{f\dot{x}}{c} = \dot{y}$. The fluent of this expression may be found in the following manner.

14. The fluxion of $x \times \overline{c^2 - x^2}|^{\frac{3}{2}}$ is $\dot{x} \times \overline{c^2 - x^2}|^{\frac{3}{2}} + \frac{3}{2} \times x \times \overline{c^2 - x^2}|^{\frac{1}{2}} \times -2x\dot{x} = \dot{x} \times \overline{c^2 - x^2}|^{\frac{3}{2}} - 3\sqrt{c^2 - x^2} \times x^2\dot{x} = \dot{x} \times \overline{c^2 - x^2}$

$\times\sqrt{c^2 - x^2} - 3\sqrt{c^2 - x^2} \times x^2\dot{x} = \overline{c^2\dot{x} - x^2\dot{x}} \times \sqrt{c^2 - x^2} - 3\sqrt{c^2 - x^2} \times x^2\dot{x} = c^2\dot{x}\sqrt{c^2 - x^2} - 4x^2\dot{x}\sqrt{c^2 - x^2}$. Consequently the fluxion of $\frac{x \times \overline{c^2 - x^2}|^{\frac{3}{2}}}{4}$ is $\dot{x}\sqrt{c^2 - x^2} \times \frac{c^2}{4} - x^2\dot{x}\sqrt{c^2 - x^2}$; and the fluxion of $\frac{2bf}{c^2} \times \frac{x \times \overline{c^2 - x^2}|^{\frac{3}{2}}}{4}$ is $\frac{2bf}{c^2} \times \dot{x}\sqrt{c^2 - x^2} \times \frac{c^2}{4} - \frac{2bf}{c^2} \times x^2\dot{x}\sqrt{c^2 - x^2}$. Hence the fluent of $\frac{2bf}{c^2} \times \dot{x}\sqrt{c^2 - x^2} \times \frac{c^2}{4} - \frac{2bf}{c^2} \times x^2\dot{x}\sqrt{c^2 - x^2} =$ the fluent of $\frac{2bf}{c^2} \times \dot{x}\sqrt{c^2 - x^2} \times \frac{c^2}{4} -$ the fluent of $\frac{2bf}{c^2} \times x^2\dot{x}\sqrt{c^2 - x^2} = \frac{2bf}{c^2} \times \frac{x \times \overline{c^2 - x^2}|^{\frac{3}{2}}}{4}$; and therefore by transposition, the fluent of $\frac{2bf}{c^2} \times \dot{x}\sqrt{c^2 - x^2} \times \frac{c^2}{4} - \frac{2bf}{c^2} \times \frac{x \times \overline{c^2 - x^2}|^{\frac{3}{2}}}{4} =$ the fluent of $\frac{2bf}{c^2} \times x^2\dot{x}\sqrt{c^2 - x^2}$. Again, as VI is equal to $\frac{2b}{c}\sqrt{c^2 - x^2}$, the fluxion of the area DVIF is $\frac{2b}{c}\sqrt{c^2 - x^2} \times \dot{y} = \frac{2bf}{c^2} \times \dot{x}\sqrt{c^2 - x^2}$, and therefore the fluent of $\frac{2bf}{c^2} \times \dot{x}\sqrt{c^2 - x^2}$ is the area DVIF; and the fluent of $\frac{2bf}{c^2} \times \dot{x}\sqrt{c^2 - x^2} \times \frac{c^2}{4}$ is the area DVIF $\times \frac{c^2}{4}$. Consequently the area DVIF $\times \frac{c^2}{4} - \frac{2bf}{c^2} \times \frac{x \times \overline{c^2 - x^2}|^{\frac{3}{2}}}{4} =$ the fluent of $\frac{2bf}{c^2} \times x^2\dot{x}\sqrt{c^2 - x^2}$, and therefore $\frac{dfg}{ac^2} \times$ area DVIF $\times \frac{c^2}{4} - \frac{dfg}{ac^2} \times \frac{2bf}{c^2} \times \frac{x \times \overline{c^2 - x^2}|^{\frac{3}{2}}}{4} =$ the fluent of $\frac{dfg}{ac^2} \times \frac{2bf}{c^2} \times x^2\dot{x}\sqrt{c^2 - x^2}$, or the force of the particles in the area DVIF to turn the ellipse. Hence, when x becomes equal to c, the force of all the particles in the semi-ellipse DGF, to turn the ellipse, is equal to the semi-ellipse DGF $\times \frac{dfg}{ac^2} \times \frac{c^2}{4} =$ the semi-ellipse DGF $\times \frac{d}{a} \times \frac{f}{4} \times g$. By article 11, therefore, the force of all the particles in the whole ellipse, which tend to turn it about the centre, is the whole ellipse PELQ $\times \frac{d}{a} \times \frac{f}{4} \times g$.

15. The other circumstances as to the figure being the same as before, let the straight line GY (Fig. 8.) touch the ellipse in G, meet QE in V and CM parallel to GH in M. Let GI be perpendicular to QE and meet it in I, and let GH meet EQ in T. Then, by a well known property of the ellipse, CI : IT : : EQ : its parameter; and therefore by the nature of a parmeter, or three proportionals, CI : IT : : $CE^2 : CP^2$, and $IT = \frac{CI \times CP^2}{CE^2}$. Hence $CT = CI - IT = CI - \frac{CI \times CP^2}{CE^2}$; and by another well known property of the ellipse, $CV = \frac{CE^2}{CI}$. Consequently $CT \times CV = CE^2 - CP^2$. Now as MAC is part of the straight line drawn from the centre of the sun to that of the earth, the angle ACE is the sun's declination, and as DC is perpendicular to AC, the angle ECD is the complement of the declination. Let $m =$ the sine of the declination, and $n =$ its cosine, and then, radius being 1, CT : CH : : 1 : m, and CV : CM or (34. I.) its equal GH : : 1 : n. By multiplication therefore, CT × CV : CH × GH : : 1 : mn. Consequenty, using the same notation as in the last article, and putting $e =$ CP, as $CT \times CV = CE^2 - CP^2$, we have $a^2 - e^2$: fg : : 1 : mn, and $fg = \overline{a^2 - e^2} \times mn$. If therefore X denote the area of the ellipse PELQ, by the last article, the force of all the particles to turn the ellipse is $\frac{d}{a} \times \frac{mn}{4} \times \overline{a^2 - e^2} \times X$.

16. Let PELQ be the same ellipse in Fig. 9, as in the 7th and 8th figures. Let the spheroid be cut by a plane parallel to PELQ, (Fig. 9,) and let the section be the ellipse HKNG; and let this ellipse be supposed to be above the plane of the paper on which PELQ is represented. Again, let the spheroid be cut by a plane passing through PL, and perpendicular to the plane PELQ, and let the line of common section of this

plane with the ellipse HKNG be HN, and let this ellipse be cut by the plane of the equator in the line KG. Then HN (14. XI.) is parallel to PL, and KG to EQ; and therefore (10. XI.) HN, KG cut one another at right angles. Again, as the axis PL is perpendicular to the plane of the equator, the plane of the equator is perpendicular (18. XI.) to the ellipse PELQ. The planes passing through the centre of the spheroid and the lines HN, KG are therefore perpendicular to the ellipse PELQ, and consequently (19. XI.) the straight line passing through the centre of the spheroid, and the point in which HN, KG cut one another, is perpendicular to HN, KG and also to PL, EQ. Consequently as the equator is a circle, KG (3. III.) is bisected by HN; and as HN is parallel to PL it is a double ordinate to the diameter of the equator passing through the point in which HN, KG cut one another, and is therefore bisected in this point. Hence, as by a well known property of the spheroid, the ellipses PELQ, HKNG are similar, it is evident that KG is the transverse, and HN the conjugate axis of the ellipse HKNG.

Let $u =$ the distance of the centre of the ellipse HKNG from the centre of the speroid, and then as the points E, K, G, Q are in the circumference of the equator, a straight line drawn from the centre of the spheroid to K or G is equal to a, and half of the straight line KG $= \sqrt{a^2 - u^2}$. Again, as the ellipses PELQ, HKNG are similar, $a : e :: \sqrt{a^2 - u^2} : \frac{e}{a}\sqrt{a^2 - u^2}$ $=$ half of HN; and $a^2 : a^2 - u^2 :: \mathrm{X} : \frac{a^2 - u^2}{a^2} \times \mathrm{X} =$ the area of the ellipse HKNG. Hence, in order to find the disturbing force of the sun on the ellipse HKNG, instead of a^2 in the expression $\frac{d}{a} \times \frac{mn}{4} \times \overline{a^2 - e^2} \times \mathrm{X}$ we are to put $a^2 - u^2$, instead of

e^2 we are to put $\frac{e^2}{a^2} \times \overline{a^2 - u^2}$, and instead of X we are to put $\frac{a^2 - u^2}{a^2}$ X. These substitutions being made, we have $\frac{d}{a} \times \frac{mn}{4} \times \frac{a^2 - e^2}{a^2} \times \overline{a^2 - u^2} \times \frac{a^2 - u^2}{a^2}$ X for the sun's disturbing force on the ellipse HKNG, tending to turn the ellipse about an axis passing through C and perpendicular to the plane PELQ. This expression for the force being multiplied by $\dot{u}$, gives the fluxion of the force on that part of the spheroid between the ellipses PELQ, HKNG; and the fluent of this, when u becomes equal to a is $\frac{d}{a} \times \frac{mn}{4} \times \frac{a^2 - e^2}{a^2} \times \frac{8a^3}{15}$ X. Consequently the double of this, *viz.* $\frac{d}{a} \times mn \times \overline{a^2 - e^2} \times \frac{4a}{15}$ X expresses the sun's disturbing force on the whole spheroid. Hence if Z $= \frac{4a}{3}$X, which expresses the solid content of the spheroid, the force on the whole spheroid is $\frac{d}{a} \times \frac{mn}{5} \times \overline{a^2 - e^2} \times$ Z. Let this be called the librating force or pressure, or the force causing libration.

17. It is evident, from the manner in which the librating pressure is calculated, that the whole of the disturbing force is occasioned by the protuberance of the spheroid above the greatest inscribed sphere. For if PELQ were a sphere, as VI (Fig. 7.) is parallel to the diameter DF, and AC perpendicular to it, the straight line VI (3. III.) would be bisected in R; and therefore the disturbing forces, above and below AC would exactly counteract one another.

Let DCF (Fig. 9.) denote a plane perpendicular to the straight line SC, then it is evident that the librating pressure tends to move the earth about that diameter of the equator, which is the common section of the equator and the plane DCF. For the sake of precision hereafter let this diameter of the equator be called the axis of libration.

The point E of the equator, nearest the sun, is at the distance of a quadrant from either extremity of the axis of libration. For, by hypothesis, SC is at right angles to the plane DCF, and therefore the axis of libration, which is in this plane, is at right angles to SC. Again, as PC the axis of the earth is at right angles to the plane of the equator, the axis of libration, which is also in the equator, is at right angles to CP. The axis of libration, therefore, being at right angles to CS, CP in the plane PELQ, is at right angles to CE in the same plane.

18. Let ADBE (Fig. 10.) be an oblate spheroid of which AB is the transverse axis, DE the conjugate axis, and C the centre. Let AKMBLH be the equator of this spheroid, and consequently at right angles to ADBE the generating ellipse. Let the spheroid be cut through DCE by a plane DMEL at right angles to ADBE, and let MCL be its common section with the equator. Then (19. XI.) MCL is at right angles to ADBE, and therefore ACM is a right angle; and as ACD is a right angle, AC is at right angles to the plane DMEL, and consequently at right angles to any plane parallel to it. Let the spheroid be cut by a plane parallel to DMEL, and let the common section of this plane with the spheroid be the ellipse FKGH. Let this ellipse cut the plane ADBE in the straight line F*r*G, and the plane of the equator in K*r*H, the point *r* being the centre of this last formed ellipse, or that point in which it meets AB. Then, by a well known property of the spheroid, the ellipses DMEL, FKGH are similar, and the area of the first is to that of the other as CA^2 or its equal CM^2 to rK^2. Put $a = AC$ or CM, $e = DC$, $x = Cr$, and the force of each particle in the spheroid being as its distance from the

plane DMEL, let v be the force of a particle at A. Let $p =$ the area of a circle whose diameter is 1. Then $4pae =$ the area of the ellipse DMEL, and as $\overline{a+x} \times \overline{a-x} = a^2 - x^2 = r\mathrm{K}^2$, $a^2 : a^2 - x^2 :: 4pae : \frac{4pe}{a} \times \overline{a^2 - x^2} =$ the area of the ellipse FKGH. Again, $a : x :: v : \frac{vx}{a} =$ the force of a particle at r, and therefore $\frac{4pev}{a^2} \times \overline{a^2 x - x^3} =$ the force of all the particles in the ellipse FKGH. Now as this force acts at r, by the property of the lever, the power of the ellipse FKGH, to turn the spheroid, either about DE or ML as an axis, is $\frac{4pev}{a^2} \times \overline{a^2 x^2 - x^4}$; and the fluxion of this force is $\frac{4pev}{a^2} \times \overline{a^2 x^2 \dot{x} - x^4 \dot{x}}$. The fluent of this, when x becomes equal to a, is $\frac{8pa^3 ev}{15}$; and the double of this, for the force of the whole spheriod, is $\frac{16pa^3 ev}{15}$. Hence it is evident that if the force of each particle in the spheroid, to cause a revolution, be as its distance from the plane DMEL, the particles on one side of this plane having a tendency to cause a revolution in one direction, and the particles on the other side of the plane having an equal tendency to cause a revolution in the same direction, then the pressure with which the spheroid is urged to revolve, either about DE or ML, is as $\frac{16pa^3 ev}{15}$. This force is equal to $\frac{av}{5}$ Z, if Z be put equal to $\frac{16pa^2 e}{3}$, the solid content of the spheroid.

19. As the librating force $\frac{d}{a} \times \frac{mn}{5} \times \overline{a^2 - e^2} \times$ Z, ascertained in article 16, and the force $\frac{av}{5}$ Z, obtained in the last article, are calculated on the same hypothesis, *viz.* that the force of a particle is as its distance from the plane DCF in Fig. 9, or the plane DMEL in Fig. 10, if they produce equal angular velocities, the spheroids in the two figures being equal in every

respect, and all other circumstances being the same, the forces themselves must be equal. Now at either of the equinoxes the other circumstances are exactly the same in the two figures. At the vernal equinox, for instance, the straight line SACB in Fig. 9. must be in the plane of the equator, and therefore the plane DCF, perpendicular to AB, at this time must pass through the poles P, L. At the equinox, therefore, the straight line SACB, and the plane DCF in Fig. 9. are justly represented by ACB, and DMEL in Fig. 10. Hence the librating force $\frac{d}{a} \times \frac{mn}{5} \times \overline{a^2-e^2} \times Z$ at the commencement of its action, at the equinox, applies to Fig. 10, and at its commencement it is equally efficacious to cause revolution about DE or ML. We are therefore enabled to compare the effect of the librating force, or the revolution it is capable of producing, at the equinox, about ML, with the diurnal revolution of the earth about DE, in the following manner.

It being admitted that each of the two forces, stated in the beginning of the article, produces the same angular velocity, then $\frac{d}{a} \times \frac{mn}{5} \times \overline{a^2-e^2} \times Z = \frac{av}{5} Z$, and therefore $d \times mn \times \frac{a^2-e^2}{a^2} = v$. But if a constant force act for a given time t, and cause the body to move on which it acts, the velocity generated from the commencement of the motion is as the force. Consequently, as v denotes the force acting on a particle at A, during the given time t, and as the forces acting on the other particles of the spheroid are proportional to their distances from the plane DMEL; the angular velocity of A, acquired in the given time t, is also accurately expressed by v. If therefore the force $\frac{av}{5} Z$ cease to act at the end of the given time t, the point A, as the spheroid is in free space, will afterwards revolve

with the uniform angular velocity v; but by the doctrine of constant forces, the angle described by A, during the action of $\frac{av}{5}Z$ is equal to $\frac{v}{2} = \frac{d}{2} \times mn \times \frac{a^2 - e^2}{a^2}$.

20. Let AB (Fig. 6.) represent that diameter of the equator about which the librating force begins to cause revolution at the equinox. Let G be the centre of the earth, and in GB let GV be taken equal to $\frac{v}{2}$, or $\frac{d}{2} \times mn \times \frac{a^2 - e^2}{a^2}$. Let w denote the angular diurnal velocity of the earth about its axis DC: and in GC let GW be taken equal to w. The points V, W being joined, let TGS be drawn parallel to VW, and by article 6, TS is the axis about which the earth will now revolve, in consequence of the diurnal revolution being combined with the libration about AB. From T, a pole of this axis, let TK be drawn perpendicular to DG. Then, by article 9, GW : GV : : GK : KT. But as the angle DGT is extremely small, GK may be considered as equal to the radius, and the arc DT as equal to its sine. Consequently, using the notation already specified, and considering a as radius, $w : \frac{d}{2} \times mn \times \frac{a^2 - e^2}{a^2} : : a : \frac{ad}{2w} \times mn \times \frac{a^2 - e^2}{a^2}$ = the angular velocity caused by the librating force. Our next object is to find the value of d in known terms.

21. If t be put for the time of the earth's diurnal revolution round its axis, and T be put for the earth's annual revolution round the sun, then $\frac{w^2}{a}$ is equal to the centripetal force on a body revolving at the equator in the time t, with the velocity w; and, using the notation of article 11, $\frac{M}{SC^2}$ is equal to the centripetal force of the sun on the earth. By the doctrine of centripetal forces therefore $\frac{M}{SC^2} : \frac{w^2}{a} : : \frac{SC}{T^2} : \frac{a}{t^2}$, and $\frac{a \times M}{SC^2 \times t^2} =$

$\frac{w^2 \times SC}{a \times T^2}$, and $M = \frac{w^2 \times SC^3 \times t^2}{a^2 \times T^2}$. But, by article 11, the disturbing force of the sun on a particle at G (Fig. 7.) is equal to $\frac{3M \times GH}{SC^3}$; and at the distance a from DF the disturbing force is $\frac{3M \times a}{SC^3}$. The foregoing value of M being substituted for it in this expression, we have $\frac{3w^2 \times t^2}{a \times T^2} = d$, the sun's disturbing force at the distance a from DF.

This value of d being put for it in the expression at the end of the last article, it follows that the angular velocity of libration, at its commencement at the equinox, is to the uniform angular diurnal velocity as $\frac{3w \times t^2}{2T^2} \times mn \times \frac{a^2 - e^2}{a^2}$ to w, or as $\frac{3t^2}{2T^2} \times mn \times \frac{a^2 - e^2}{a^2}$ to 1. But, according to the preceding notation, $t : \dot{t} :: 360^\circ : \frac{360\dot{t}}{t}$ = the uniform angular diurnal velocity, and therefore $1 : \frac{3t^2}{2T^2} \times mn \times \frac{a^2 - e^2}{a^2} :: \frac{360\dot{t}}{t} : 360 \times \frac{3t\dot{t}}{2T^2} \times mn \times \frac{a^2 - e^2}{a^2}$ = the angular velocity of libration, at its commencement at the equinox. But as the product mn is the only variable quantity which enters into the value of the librating force, obtained in article 16, it is evident that $360 \times \frac{3t\dot{t}}{2T^2} \times mn \times \frac{a^2 - e^2}{a^2}$ expresses the momentary angular velocity of libration at any time. We are now to consider this effect of the librating force in the direction in which the force is exerted, *viz.* in a meridian analogous to PEAQ in Figure 7.

22. Let FLGA (Fig. 11.) represent the ecliptic on the sphere, S the sun's place in it, L the first point of Libra and A that of Aries; LBA the position of the equator when the sun is at S, and SB the sun's declination. Let FBG be the position into which the equator is pressed in the time $\dot{t}$, by a combination of the librating force and the diurnal

revolution; or, which comes to the same, let the spherical angle FBL or ABG be equal to $360 \times \frac{3t\dot{t}}{2T^2} \times mn \times \frac{a^2 - e^2}{a^2}$. Let $h = \frac{a^2 - e^2}{a^2}$. Then by spherical Trigonometry, sin. F : sin. BL : : sin. B : sin. FL; and therefore, as FL and the angle FBL are extremely small, the momentary precession $FL = 360 \times \frac{3t\dot{t}}{2T^2} \times hmn \times \frac{\text{sin. BL}}{\text{sin. F}}$.

23. If FG be bisected in C, and the arc CE be perpendicular to FG, meeting LBA in E, and FBG in D; then the arc CD is the measure of the angle at F. Also, as FL is extremely small, CE may be taken for the measure of the angle at L. Hence as BDE is a right angle, radius : sin. BE or cos. BL : : sin. EBD : sin. ED. Consequently as ED is extremely small, $360 \times \frac{3t\dot{t}}{2T^2} \times hmn \times \frac{\text{cos. BL}}{\text{radius}} = \text{ED}$, the momentary nutation, or the momentary change in the inclination of the equator to the ecliptic.

From the last proportion, and that concluding the preceding article, it follows that FL : ED : : $\frac{\text{sin. BL}}{\text{sin. F}} : \frac{\text{cos. BL}}{\text{radius}} : : \frac{\text{radius} \times \text{sin. BL}}{\text{cos. BL}}$: sin. F. But cos. BL : radius : : sin. BL : $\frac{\text{radius} \times \text{sin. BL}}{\text{cos. BL}} = \text{tang. BL}$; and therefore, the momentary precession FL is to the momentary nutation ED, as the tangent of the right ascension BL to the sine of the obliquity of the ecliptic.

24. Let $b =$ the sine of the obliquity of the ecliptic, $c =$ its cosine; $z =$ the arc LS, $x =$ its sine, and $y =$ its cosine; and let $2p =$ the circumference of the ecliptic. Then as LBS is a right angle, by the circular parts, cos. BS $\times$ cos. BL $=$ radius $\times$ cos. LS; that is $n \times$ cos. BL $= y$, radius being 1. Again, radius : x : : b : $bx =$ the sine of BS $= m$. Consequently, cos. BS $\times$ sin. BS $\times$ cos. BL $= mn \times$ cos. BL $= bxy$; and therefore,

by the preceding article, the momentary nutation is $360 \times \frac{3t\dot{t}}{2T^2} \times hbxy$, radius being 1.

Again, $2p : \dot{z} :: T : \frac{T\dot{z}}{2p} = \dot{t}$. Also $\sqrt{1-xx} = y$, and, by the fluxional doctrine of circular arcs, $\dot{z} = \frac{\dot{x}}{\sqrt{1-xx}}$; and therefore $\dot{t} = \frac{T\dot{x}}{2p\sqrt{1-xx}}$. These values of $\dot{t}$ and y being put for them in the above expression, the momentary nutation, or, which is the same thing, the fluxion of the nutation is $360 \times \frac{3t}{2T} \times \frac{hbx\dot{x}}{2p}$. Consequently the nutation, when the sun is at S, is $360 \times \frac{3t}{4T} \times \frac{hbxx}{2p}$.

When the sun arrives at the solstitial point C, then x becomes equal to 1, and the nutation is then $360 \times \frac{3t}{4T} \times \frac{hb}{2p} = 180 \times \frac{3t}{4T} \times \frac{a^2-e^2}{a^2} \times \frac{b}{p}$ in degrees, or $10800 \times \frac{3t}{4T} \times \frac{a^2-e^2}{a^2} \times \frac{b \times 60}{p}$ in seconds. Now $t =$ one sidereal day, $T = 366\frac{1}{4} = \frac{1465}{4}$, and therefore $4T = 1465$. According to Sir ISAAC NEWTON'S determination of the figure of the earth, a is as 231, e as 230, and therefore $\frac{a^2-e^2}{a^2} = \frac{461}{53361}$. Also supposing the obliquity of the ecliptic 23° 27′ 45″, $b = .3981487$, and $p = 3.14159265$. Consequently $10800 \times \frac{3t}{4T} \times \frac{a^2-e^2}{a^2} \times \frac{b \times 60}{p} = 10800 \times \frac{3}{1465} \times \frac{461}{53361} \times \frac{23.888922}{3.14159265}$, the computation of which may be finished in the following manner.

10800	Log. 4.0334238	1465	Log. - - 3.1658376
3	Log. 0.4771213	53361	Log. - - 4.7272240
461	Log. 2.6637009	3.14159265	Log. 0.4971499
23.888922	Log. 1.3781998		8.3902115
	8.5524458		
	8.3902115		
	0.1622343 = Log. of 1″.4529, and there-		

fore the nutation caused by the action of the sun in a quarter of a year is 1″ 27‴ nearly.

25. By the circular parts, radius $\times c =$ cot. LS $\times$ tang. BL $= c$, radius being 1. But $x : \sqrt{1-xx} :: 1 : \frac{\sqrt{1-xx}}{x} =$ cot. LS; and therefore $\frac{cx}{\sqrt{1-xx}} =$ tang. BL. Consequently by the last and article 23, $b : \frac{cx}{\sqrt{1-xx}} :: 360 \times \frac{3t}{2T} \times \frac{bbx\dot{x}}{2p} : 360 \times \frac{3ct}{4T} \times \frac{b}{p} \times \frac{x^2 \dot{x}}{\sqrt{1-xx}} =$ the fluxion of the precession when the sun is at S. Now the fluent of $\frac{x^2 \dot{x}}{\sqrt{1-xx}}$ is $\frac{z - x\sqrt{1-xx}}{2}$. For $\dot{z} = \frac{\dot{x}}{\sqrt{1-xx}}$, and the fluxion of $x\sqrt{1-xx}$ is $\dot{x}\sqrt{1-xx} - \frac{x^2 \dot{x}}{\sqrt{1-xx}}$. Consequently the whole fluxion of $z - x\sqrt{1-xx}$ is $\frac{\dot{x}}{\sqrt{1-xx}} - \dot{x}\sqrt{1-xx} + \frac{x^2 \dot{x}}{\sqrt{1-xx}} = \frac{\dot{x}}{\sqrt{1-xx}} - \frac{\dot{x}}{\sqrt{1-xx}} + \frac{x^2 \dot{x}}{\sqrt{1-xx}} + \frac{x^2 \dot{x}}{\sqrt{1-xx}} = \frac{2x^2 \dot{x}}{\sqrt{1-xx}}$. Consequently the fluent of the precession, when the sun is at S, is $360 \times \frac{3ct}{4T} \times \frac{b}{p} \times \frac{z - x\sqrt{1-xx}}{2}$.

When the sun arrives at the solstitial point C, then x becomes equal to 1, and z becomes equal to $\frac{p}{2}$, and the quantity of the precession is then $360 \times \frac{3ct}{4T} \times \frac{b}{p} \times \frac{p}{4} = 360 \times \frac{3t}{4T} \times \frac{cb}{4}$. This expressed in numbers, admitting the obliquity of the ecliptic to be 23° 27′ 45″, is $90 \times \frac{3}{1465} \times \frac{461}{53361} \times .9173813$ in degrees, and the same in seconds is $5400 \times \frac{3}{1465} \times \frac{461}{53361} \times 55.042878$. This calculation may be finished in the following manner.

5400	Log. 3.7323938	1465 -	Log. 3.1658376
3	Log. 0.4771213	53361 -	Log. 4.7272240
461	Log. 2.6637009		7.8930616
55.042878	Log. 1.7407010		
	8.6139170		
	7.8930616		
	0.7208554 = Log. of 5″.2584.		

Consequently the annual precession, caused by the disturbing force of the sun, is 21″.0336.

The obliquity of the ecliptic has been assumed as equal to 23° 27′ 45″, such being its magnitude, very nearly, at the beginning of the year 1807.

From the general expression $360 \times \frac{3t}{4T} \times \frac{bbxx}{2p}$, obtained in article 24, it is evident, that when the sun is in either of the equinoctial points, the nutation becomes equal to o. Supposing therefore the earth to be subject to no other disturbing force than that of the sun, at each of the equinoxes the earth's diurnal revolution is made about its axis of figure, as PL in Fig. 9; but as at other times the disturbing force tends to cause a libration about a diameter of the equator, it is evident from article 10, that the axis about which it revolves deviates, by a quantity extremely small, from its axis of figure. A similar deviation, of the axis of revolution from the axis of figure, is produced by the action of the moon; but a minute examination of these deviations is foreign to the present design. As the foregoing articles extend beyond the supposed difficulty in the subject, it is deemed unnecessary either to add to their number, or to lengthen this Paper by such additional remarks, as may be met with in every respectable publication on Physical Astronomy.

Fig. 2.

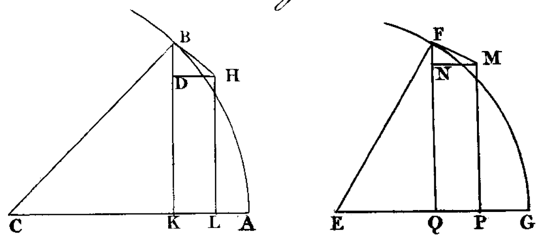


Fig. 7.

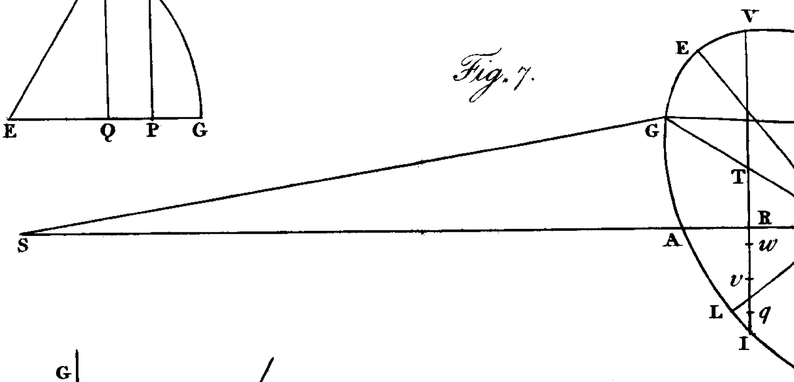


Fig. 3.

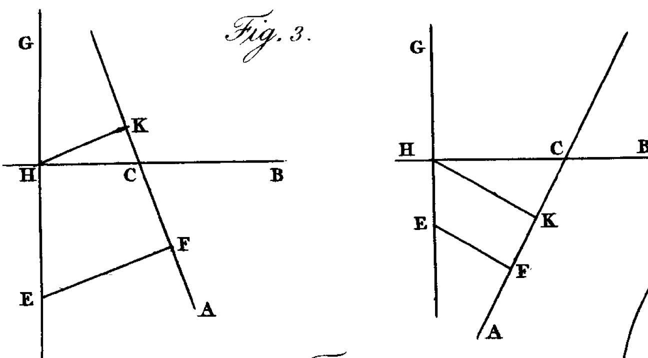


Fig. 6.

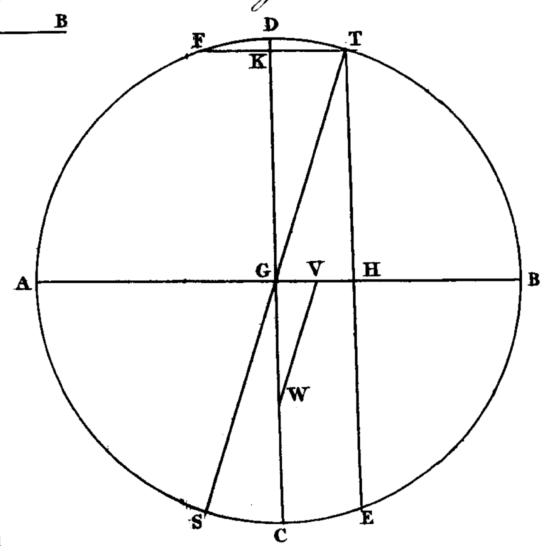


Fig. 4.

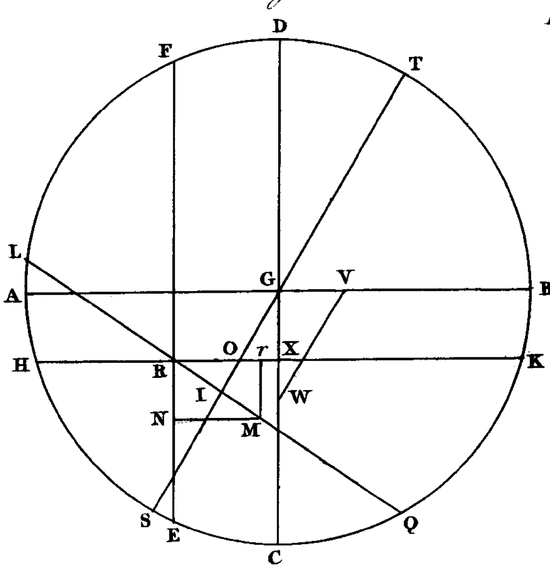
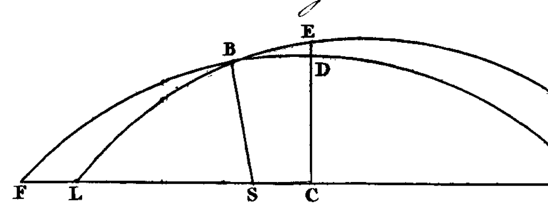
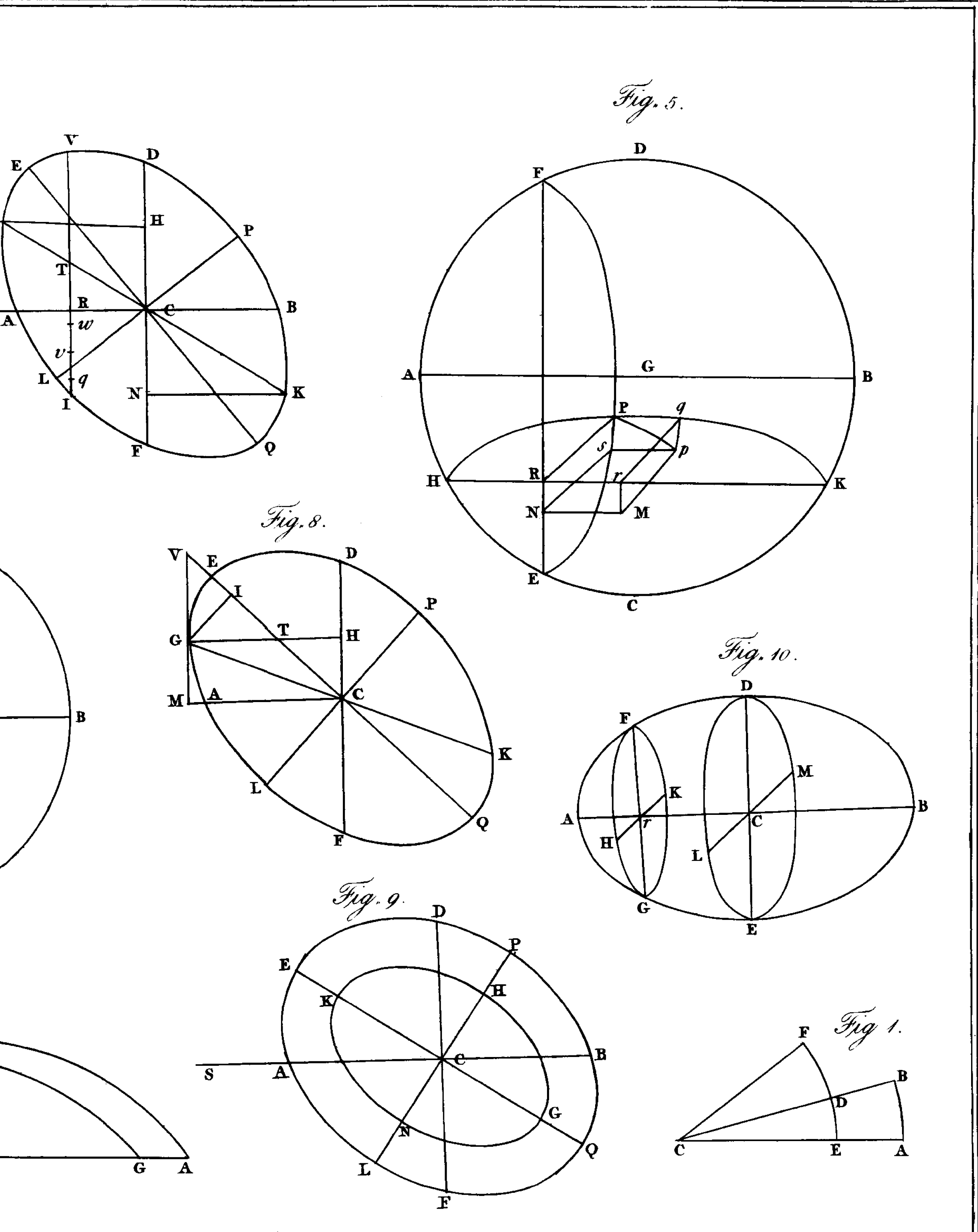


Fig. 11.





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